Streamline Computations Available in FEFLOW

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1 INTRODUCTION

The evaluation of velocity fields is of important interest in the finite-element flow analysis. Commonly, velocities $\boldsymbol{v}$ are derived as nodal quantities from primary variables such as hydraulic head h or pressure p by using suited projection (smoothing) techniques as described elsewhere[1,2,3]. If velocity $\boldsymbol{v}$ is known different evaluation methods are available to trace and visualize the flow field in postprocessing procedures. The most general method concerns particle tracking[4], where a *pathline* of an individual fluid particle is traced in space $\boldsymbol{x}$ and time t via a Lagrangian approach. Particle tracking methods are applicable in FEFLOW both in two dimensions (2D) and three dimensions (3D) under very general flow conditions (presence of interior sinks/sources and/or boundary conditions such as pumping wells and others). While they refers to individually moving particles which have to be appropriately assigned at starting positions, a continuous picture of the overall flow movement is sometimes difficult to attain, even if a large number of particles are traced.

There are efficient, but specific alternative methods for limited cases in 2D applications. These methods represent *streamline* integrators, which facilitate the computation of flow pattern and distributed discharge through the flow systems in a direct way. The theoretical basis of the two most important streamline integrators, which are implemented in FEFLOW, will be described in the following.

2 PRELIMINARIES

We consider both Cartesian and cylindrical coordinate systems, such as

$$\boldsymbol{x} = \begin{Bmatrix} x, y, z \\ x, y \\ r, \vartheta, z \end{Bmatrix} \quad \text{for} \quad \begin{cases} \text{3D} \\ \text{2D} \\ \text{axisymmetry} \end{cases} \tag{1}$$

where x, y are Cartesian coordinates, z is vertical or axial coordinate, r is radial coordinate and ϑ is azimuthal angle. The velocity vector $\boldsymbol{v}$ is accordingly defined as

$$\boldsymbol{v} = \begin{cases} \begin{bmatrix} u \\ v \\ w \end{bmatrix} & \text{for} \quad \text{Cartesian coordinates} \\ \begin{bmatrix} v_r \\ v_\vartheta \\ v_z \end{bmatrix} & \qquad\ \ \text{cylindrical coordinates} \end{cases} \tag{2}$$

The scalar product $\nabla \cdot \boldsymbol{v}$ is given by

$$(\nabla \cdot \boldsymbol{v}) = \begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} & \text{3D } (x, y, z) \text{ Cartesian} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} & \text{2D } (x, y) \text{ Cartesian} \\ \dfrac{1}{r}\dfrac{\partial (r v_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial v_\vartheta}{\partial \vartheta} + \dfrac{\partial v_z}{\partial z} & \text{cylindrical } (r, \vartheta, z) \end{cases} \tag{3}$$

The vector cross-product $(\nabla \times \boldsymbol{v})$ reads

$$(\nabla \times \boldsymbol{v}) = \begin{cases} \begin{bmatrix} \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{bmatrix} & \text{3D } (x, y, z) \text{ Cartesian} \\ \begin{bmatrix} 0 \\ 0 \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{bmatrix} & \text{2D } (x, y) \text{ Cartesian} \\ \begin{bmatrix} \dfrac{1}{r}\dfrac{\partial v_z}{\partial \vartheta} - \dfrac{\partial v_\vartheta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} - \dfrac{\partial v_r}{\partial z} \\ \dfrac{1}{r}\dfrac{\partial (r v_\vartheta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial v_r}{\partial \vartheta} \end{bmatrix} & \text{cylindrical } (r, \vartheta, z) \end{cases} \tag{4}$$

In other notation $(\nabla \cdot \boldsymbol{v})$ is called the *divergence* of velocity vector $\boldsymbol{v}$

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} \tag{5}$$

and $(\nabla \times \boldsymbol{v})$ is called the curl of velocity vector $\boldsymbol{v}$ or the *vorticity vector* $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v} = \nabla \times \boldsymbol{v} \tag{6}$$

From (4) is can be recognized that $\boldsymbol{\omega}$ in 3D represents a general vector field. Contrarily, in 2D and for axisymmetric situations if assuming that all dependencies with respect to the azimuthal direction ϑ vanish, i.e., $v_\vartheta = \partial / \partial \vartheta = 0$, we find the following useful curl-relations

$$\omega = \|\boldsymbol{\omega}\| = \begin{cases} \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} & \text{2D } (x, y) \text{ Cartesian} \\ \dfrac{\partial v_z}{\partial r} - \dfrac{\partial v_r}{\partial z} & \text{axisymmetric } (r, z) \end{cases} \tag{7}$$

where ω represents the (scalar) *vorticity* function.

3 STREAMLINES AND STREAMFUNCTION

A *streamline* is the locus of points that are everywhere tangent to the instantaneous velocity vector $\boldsymbol{v}$. If $\boldsymbol{ds}$ is an element of length along a streamline, and thus tangent to the local velocity vector, then the equation of a streamline is given by (Fig. 1)

$$\boldsymbol{ds} \times \boldsymbol{v} = 0 \tag{8}$$

or, in 2D Cartesian coordinates

$$\frac{dx}{u} = \frac{dy}{v} \tag{9}$$

Two streamlines cannot intersect except where $\boldsymbol{v} = 0$.

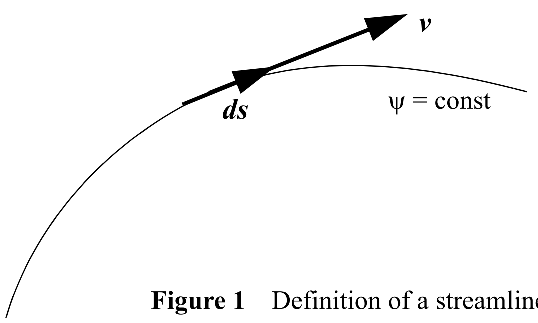

Figure 1 Definition of a streamline.

Since, by definition, no flow can cross a streamline it requires that the velocity vector field $\boldsymbol{v}$ have to be divergence-free (solenoidal)

$$\nabla \cdot \boldsymbol{v} = 0 \tag{10}$$

That means the flow is to be steady-state and no distributed sources and sinks can exist in the flow domain.

An equation that would describe such streamlines in a 2D (and axisymmetric) flow may be written in the form

$$\psi = \psi(x, y) \tag{11}$$

where ψ is called the *streamfunction*. When ψ is constant (11) describes a streamline. It must obey the general differential relation for the change in the streamfunction, $d\psi$,

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \tag{12}$$

The following definition relates ψ and the velocity components

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} \tag{13}$$

for 2D and

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial z} \qquad v_z = -\frac{1}{r}\frac{\partial\psi}{\partial r} \tag{14}$$

for axisymmetric flow. The definitions (13) and (14) automatically satisfy the condition of free divergence (10) in using (3). Substituting (13) into (12) it gives

$$d\psi = -v\ dx + u\ dy \tag{15}$$

A major characteristic of the streamfunction is that the difference in ψ between two streamlines is equal to the volume flow rate Q between those streamlines. Let us consider two streamlines with values ψ_A and ψ_B as shown in Fig. 2.

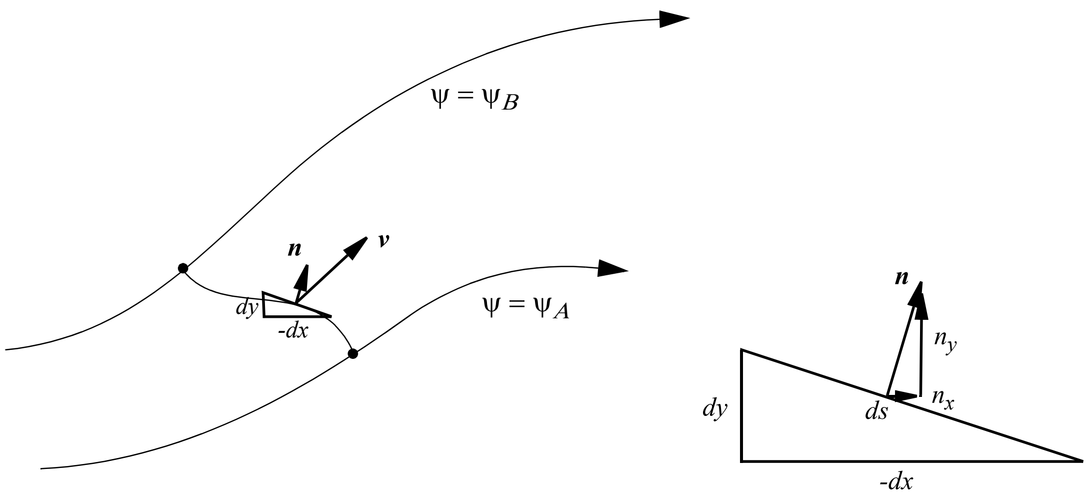

Figure 2 Streamfunction in a plane flow.

The volume flow rate between the streamlines is

$$Q_{AB} = \int_A^B \boldsymbol{v}\cdot\boldsymbol{n}\ ds = \int_A^B (un_x + vn_y)ds \tag{16}$$

where $\boldsymbol{n}$ is the normal unit vector. By geometry we have the relations $n_x ds = dy$ and $n_y ds = -dx$. With these relations (16) becomes

$$\begin{aligned} Q_{AB} &= \int_A^B (u dy - v dx) = \int_A^B d\psi \\ Q_{AB} &= \psi_B - \psi_A \end{aligned} \tag{17}$$

The flow rate between streamlines is the difference in their streamfunction values. This equation is also unaffected by the addition of an arbitrary constant to ψ.

4 STREAMLINE INTEGRATOR BY USING VORTICITY EQUATION

For 2D and axisymmetric flows a very efficient approach to computing the streamfunction distribution for a given velocity field is based on using the vorticity function ω. By substituting the streamfunction definition (13) into the vorticity equation (7) the following elliptic partial differential (Poisson) equation is obtained

$$-\nabla^2\psi = \omega \tag{18}$$

or

$$-\nabla^2\psi = \begin{cases} \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} & \text{2D } (x, y) \text{ Cartesian} \\ \left(\dfrac{\partial v_z}{\partial r} - \dfrac{\partial v_r}{\partial z}\right) r & \text{axisymmetric } (r, z) \end{cases} \tag{19}$$

Equation (19) can be easily solved by the finite element method if formulating an unique boundary value problem of the domain Ω enclosed by the boundary Γ. The Galerkin-based finite element formulation of (19) gives (exemplified for 2D Cartesian)

$$\int_\Omega (\nabla \boldsymbol{N} \nabla \boldsymbol{N}^T d\Omega)\boldsymbol{\Psi} = \int_\Omega \left(\frac{\partial \boldsymbol{N}^T}{\partial x}\boldsymbol{V} - \frac{\partial \boldsymbol{N}^T}{\partial y}\boldsymbol{U}\right) d\Omega + \int_\Gamma \boldsymbol{N}(\boldsymbol{n} \cdot \nabla\psi) d\Gamma \tag{20}$$

by introducing finite element interpolation functions for the streamfunction and velocity components

$$\begin{aligned} \psi &= \boldsymbol{N}^T \boldsymbol{\Psi} \\ u &= \boldsymbol{N}^T \boldsymbol{U} \\ v &= \boldsymbol{N}^T \boldsymbol{V} \end{aligned} \tag{21}$$

where $\boldsymbol{\Psi}, \boldsymbol{U}, \boldsymbol{V}$ are nodal vectors and $\boldsymbol{N}$ are finite element shape functions. The boundary integral in (20) vanishes because the flux normal to the streamline direction is zero, $\boldsymbol{n} \cdot \nabla\psi = n_x \frac{\partial \psi}{\partial x} + n_y \frac{\partial \psi}{\partial y} = -n_x v + n_y u = 0$ if the velocity vector field $\boldsymbol{v}$ is divergence-free (solenoidal), i.e., $\nabla \cdot \boldsymbol{v} = 0$. Accordingly, the following linear matrix system results

$$\boldsymbol{A} \cdot \boldsymbol{\Psi} = \boldsymbol{B}(\boldsymbol{U}, \boldsymbol{V}) \tag{22}$$

for solving the streamfunction $\boldsymbol{\Psi}$ at each nodal point of a finite element mesh with known nodal velocity components $\boldsymbol{U}, \boldsymbol{V}$. The matrix $\boldsymbol{A}$ is symmetric and sparse. The equation system (22) is solved via standard matrix solvers. However, a suitable boundary condition for $\boldsymbol{\Psi}$ is required. Practically, at only one node on the outer boundary Γ the streamfunction is set to a Dirichlet-type reference value of zero. The solution (22) is restricted to a solenoidal 2D (or axisymmetric) velocity vector field $\boldsymbol{v}$, i.e., steady-state flow, no interior boundary conditions (e.g., fluxes, wells) and absence of sinks and sources.

5 STREAMLINE INTEGRATOR BY USING BOUNDARY INTEGRAL

This streamline integration method is based on the numerically solution of differential (16) written in the form

$$\delta\psi = \int_A^B (un_x + vn_y)ds \tag{23}$$

where $\delta\psi$ is the change in the streamfunction which is to be solved along a defined boundary. In the preferred method the computation of $\delta\psi$ is carried out using (23) along each boundary of finite elements, where the integration path *AB* is taken as element edge. We consider a typical element boundary as shown in Fig. 3. The following finite element interpolations are introduced

$$\begin{aligned} u &= \boldsymbol{N}^T\boldsymbol{U} \qquad & x &= \boldsymbol{N}^T\boldsymbol{X} \\ v &= \boldsymbol{N}^T\boldsymbol{V} \qquad & y &= \boldsymbol{N}^T\boldsymbol{Y} \end{aligned} \tag{24}$$

where $\boldsymbol{N}$ are finite element shape functions and $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{X}, \boldsymbol{Y}$ are vectors of nodal point velocities and coordinates.

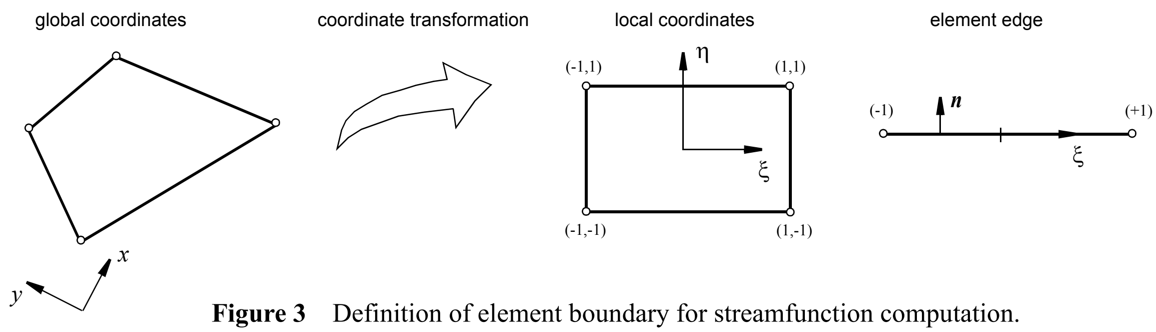

Figure 3 Definition of element boundary for streamfunction computation.

In the finite element standard procedure the global coordinates (x, y) in 2D are transformed to local coordinates (ξ, η) (Fig. 3). For a infinitesimal line element ds it results

$$ds = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} d\xi = L\, d\xi \tag{25}$$

written for the local coordinate $(-1 \le \xi \le 1)$ at element boundaries where $\eta = \pm 1$. In (25) L corresponds to the length of the boundary segment. Using (25) we find

$$\frac{\partial s}{\partial x} = L\frac{\partial \xi}{\partial x} \qquad \frac{\partial s}{\partial y} = L\frac{\partial \xi}{\partial y} \tag{26}$$

and their inverse

$$\frac{\partial x}{\partial s} = \frac{1}{L}\frac{\partial x}{\partial \xi} \qquad \frac{\partial y}{\partial s} = \frac{1}{L}\frac{\partial y}{\partial \xi} \tag{27}$$

Using (27) the unit normal vector can be expressed by

$$\boldsymbol{n} = \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} \frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial s} \end{bmatrix} = \frac{1}{L}\begin{bmatrix} \frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \xi} \end{bmatrix} \tag{28}$$

Combining (25) and (28) with (23) the streamline integral along any finite element boundary takes the form

$$\delta\psi = \int_{-1}^{+1}\left(\frac{\partial \boldsymbol{N}^T}{\partial \xi}\boldsymbol{Y}\,\boldsymbol{N}^T\boldsymbol{U} - \frac{\partial \boldsymbol{N}^T}{\partial \xi}\boldsymbol{X}\,\boldsymbol{N}^T\boldsymbol{V}\right)d\xi \tag{29}$$

where the coordinate interpolation functions (24) are applied.
The change in the streamfunction along any element edges is computed from (29) with known nodal velocity vectors $\boldsymbol{U}$, $\boldsymbol{V}$. The computation of the streamfunction for an entire finite element mesh is generated by applying (29) along successive element boundaries, starting at a node for which a reference value of ψ has been specified. Unlike the vorticity integration method the present boundary integral method is only an element-by-element procedure, which is computationally efficient and does not require the solution of a matrix problem. However, the boundary integrator is also limited to solenoidal velocity fields, i.e, steady-state flow, no interior boundary conditions (e.g., fluxes, wells) and absence of sinks and sources.

6 CONCLUSIONS

FEFLOW provides two streamline integration methods (vorticity equation integrator and boundary integral method) as additional tools to evaluate velocity fields in an alternative way in contrast to particle tracking techniques. However, these streamline integrators are limited to steady-state 2D (or axisymmetric) flow problems, where neither interior boundary conditions (such as fluxes or pumping wells) nor sinks and sources should exist. While the boundary integral method does not require the solution of a matrix problem, the vorticity equation integrator produces often more accurate results and has shown more robust. Accordingly, the vorticity equation integrator represents the method of our first choice. Streamline integrators are very useful for instance in density-variable flow simulations (see 2D streamline patterns as shown in the references[1,2,3]), where complex recirculating flow patterns (eddies) can occur, which cannot be easily detected and visualized by using particle tracking methods. In such cases, although the density-coupled mass (or heat) transport process is transient, the flow field is divergence-free at each time step (FEFLOW runs in the so-called steady flow-transient transport time mode) because the absence of storage (by fluid and skeleton compression) in the flow problem assuming no interior boundary conditions and sinks/sources.

7 REFERENCES

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